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RESEARCH MEMORANDUM

SOME CHARACTERISTICS OF THE ELLIPTIC
GAUSSIAN DISTRIBUTION

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SUMMARY

The distribution of the point of impact of bombs, missiles, and other projectiles frequently is approximately Gaussian (normal) in character. If the dispersion is equal along any two perpendicular axes, then the impact probability law is termed circular Gaussian; if the dispersion is not equal along perpendicular axes, then the probability law is called elliptical Gaussian. We consider first the evaluation of the integral of an elliptical Gaussian over a circle of radius R, i.e., the probability of a missile landing within a circle of radius R if aimed at the center and subjected to an elliptical Gaussian impact probability distribution.

An equivalent problem is shown to be the evaluation of the integral of a circular Gaussian over an ellipse. This integral, termed the elliptic coverage function, is expressed as the difference of two circular coverage functions; these have been tabulated in RM-330. A short table of the elliptic coverage function is obtained and presented in a convenient graphical form. Finally, some characteristics of the circular probable error and the quartiles of an elliptic Gaussian are presented in graphical form.

PREFACE

Notes concerning the integral of an elliptic Gaussian distribution over a circle, taken by the author from lectures of H. H. Germond at the University of Florida in 1947, form the basis of this research memorandum. The present work extends the results given in RM-330, The Circular Coverage Function, by H. H. Germond.

Although this work was essentially completed in the early 1950's, there was insufficient interest in the topic at the time to justify the effort required to put it in shape for external distribution. The past two years, however, have seen a general revival of interest in Gaussian impact distributions of various types. Possibly this reflects a renewed attention to methods for estimating the kill probabilities of hardened point targets, now that city populations may no longer be the target of interest. Since there have been numerous requests for any RAND work on the topic of this research memorandum, publication now seemed desirable.

CONTENTS

SOME CHARACTERISTICS OF THE ELLIPTIC GAUSSIAN DISTRIBUTION

1. INTRODUCTION

The distribution of the points of impact of bombs, missiles, and other projectiles frequently is approximately Gaussian ("normal") in character. In general, the dispersion along any two mutually perpendicular axes will not be equal and the correlation coefficient will not be zero. Given a map showing the points of impact, it is convenient to use the mean point of impact as the origin and to choose the orientation of the axes to minimize the absolute value of the correlation coefficient. It is easily shown (see Appendix) that this choice of orientation in the case of a Gaussian distribution is equivalent to choosing axes oriented in the directions of the maximum and minimum dispersion, respectively, and that the corresponding correlation coefficient is zero. (See Refs. 1, 2.)

Without loss of generality, then, we can thus restrict attention to an uncorrelated Gaussian distribution in the xy-plane with components of standard deviation $\sigma_x \geq \sigma_y$ and the origin at the mean impact point. We designate such a distribution as an elliptical Gaussian. (Equi-probability curves are ellipses with major and minor axes aligned with the x and y axes, respectively.) We consider here the problem of evaluating the integral of such a probability function over a circle of radius R (the probability of a missile landing within a circle of radius R if aimed at the center when the missile is subject to a Gaussian impact-probability law). An equivalent problem is

the integration of a symmetric Gaussian distribution over an ellipse. We consider also the inverse problem of determining for a given elliptic Gaussian distribution the radius R_q of that circle which includes 100q per cent of the impacts (that circle over which the integral of the Gaussian is q). Of particular interest are the 0.5 circle, having radius $R_{0.5}$ equal to the "circular probable error" (CEP), and the upper and lower circular quartiles, $R_{0.25}$ and $R_{0.75}$. We consider the relation of these radii to the standard deviations σ_x and σ_y and also to the arithmetic and quadratic means of the σ's.

2. THE CIRCULAR COVERAGE FUNCTION

We include here the definition and some characteristics of the circular coverage function that will be needed later. In Ref. 1, Germond designates the integral of a circular (symmetric) Gaussian distribution with unit standard deviation over a circle of radius R and at a distance r from the origin as $p(R,r)$, the circular coverage function. Thus $p(R/\sigma, r/\sigma)$ is the probability that a missile will hit within a circle of radius R if aimed at a point set off a distance r from the center of the circle, where the missile is subject to the Gaussian impact—probability law with zero correlation and symmetric standard deviations $\sigma = \sigma_x = \sigma_y$. The function $p(R,r)$ is expressible in the following equivalent forms:

$$(2.1) \qquad p(R,r) = \exp(-r^2/2) \int_0^R \eta \exp(-\eta^2/2) \, I_0(r\eta) \, d\eta$$

$$(2.2) \qquad = 1 - \exp(-r^2/2) \int_R^\infty \eta \exp(-\eta^2/2) \, I_0(r\eta) \, d\eta$$

$$(2.3) \qquad = \exp[-(R^2 + r^2)/2] \sum_{n=1}^\infty (R/r)^n \, I_n(Rr)$$

$$(2.4) \qquad = 1 - \exp[-(R^2 + r^2)/2] \sum_{n=0}^\infty (r/R)^n \, I_n(Rr),$$

where $I_n(z)$ is the modified Bessel function of the first kind, of order n. Tables of $p(R,r)$ have been prepared in Ref. 1.

3. THE ELLIPTIC COVERAGE FUNCTION

We designate the integral of a symmetric Gaussian distribution of unit standard deviation and zero correlation coefficient over an ellipse with major axis α and minor axis β as $q(\alpha,\beta)$, the "elliptic coverage function." Thus $q(\alpha/\sigma, \beta/\sigma)$ is the probability that a missile will impact within the ellipse (α,β) if aimed at the center when the missile is subject to a Gaussian impact—probability law with zero correlation and symmetric standard deviations σ. We show that it is expressible as the difference of two circular coverage functions, and is thus also a tabulated function. Further, we show that the integration of an elliptic Gaussian over a circle is an equivalent problem. Finally, using the tabulated values of the circular coverage function, we present a simplified graphical representation of the function $q(R/\sigma_y, R/\sigma_x)$.

We first prove three lemmas that will later be needed.

Lemma 1. For μ real, $|\mu| < 1$, we have

$$(3.1) \quad \frac{1 - \mu^2}{1 + \mu^2 + 2\mu \cos \varphi} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \mu^n \cos (n\varphi).$$

Proof. Consider the complex variable $z = \lambda e^{i\varphi}$. For $\lambda < 1$, the function $(1 - z)/(1 + z)$ has the power series expansion

$$\frac{1 - z}{1 + z} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \lambda^n e^{in\varphi} .$$

Taking real values of both sides, we obtain

$$\frac{1 - \lambda^2}{1 + \lambda^2 + 2\lambda \cos \varphi} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \lambda^n \cos (n\varphi).$$

Thus the lemma holds for μ positive. For μ negative, the proof is similar, involving now $(1 + z)/(1 - z)$ in place of $(1 - z)/(1 + z)$.

Lemma 2. The function $e^{z \cos \varphi}$ for all z and φ may be expressed by the expansion

$$(3.2) \quad e^{z \cos \varphi} = I_0(z) + 2 \sum_{n=1}^{\infty} I_n(z) \cos (n\varphi),$$

where I_n is the modified Bessel function of the first kind, of order n.

Proof. This well-known expansion in Bessel functions is most easily proved in terms of the Bessel coefficients. It may be verified from the definitions of Bessel functions that,

for all z and nonzero values of t,

$$\exp\left[\frac{z}{2}(t + t^{-1})\right] = \sum_{n=-\infty}^{\infty} I_n(z)t^n.$$

Letting $t = e^{i\varphi}$, we see that

$$\exp(z \cos \varphi) = \sum_{n=-\infty}^{\infty} I_n(z)e^{in\varphi}.$$

Equating real parts and noting that $I_{-n}(z) = I_n(z)$, we get

$$e^{z \cos \varphi} = I_0(z) + 2 \sum_{n=1}^{\infty} I_n(z) \cos (n\varphi).$$

Lemma 3. If $p(R,r)$ is the circular coverage function of Sec. 2, then

(3.3) $$p(R,r) - p(r,R) = 1 - \exp[- (R^2 + r^2)/2]$$

$$\cdot \left[I_0(rR) + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n I_n(Rr)\right].$$

Proof. Using the form (2.3) for $p(r,R)$ and (2.4) for $p(R,r)$, we obtain

$$p(R,r) - p(r,R) = 1 - \exp[-(R^2 + r^2)/2] \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n I_n(Rr)$$

$$- \exp[- (R^2 + r^2)/2] \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n I_n(Rr),$$

whence (3.3) follows.

Theorem 1. The elliptic coverage function $q(A,B)$, the integral of a symmetric Gaussian distribution of unit standard deviation over an ellipse with major axis A and minor axis B, is given by the difference of two circular coverage functions

$$(3.4) \qquad q(A,B) = p\left(\frac{A+B}{2}, \frac{A-B}{2}\right) - p\left(\frac{A-B}{2}, \frac{A+B}{2}\right).$$

Proof. By definition,

$$q\left(\frac{\alpha}{\sigma}, \frac{\beta}{\sigma}\right) = \frac{1}{2\pi\sigma^2} \int\!\!\int_C \exp[-(x^2 + y^2)/2\sigma^2]dx\,dy,$$

where the integrations are over the area bounded by the curve $C: x^2/\alpha^2 + y^2/\beta^2 = 1$. Letting $A = \alpha/\sigma$ and $B = \beta/\sigma, \xi = x/\sigma$ and $\eta = y/\sigma$, we obtain

$$(3.5) \qquad q(A,B) = \frac{1}{2\pi} \int\!\!\int_{C'} \exp[-(\xi^2 + \eta^2)/2]\,d\xi\,d\eta$$

where $C': \xi^2/A^2 + \eta^2/B^2 = 1$.

Making a transformation to polar coordinates, $\xi = \rho \cos\theta$, $\eta = \rho \sin\theta$, and letting

$$L^2 = \left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right)^{-1},$$

from (3.5) we obtain

$$(3.6) \qquad q(A,B) = \frac{1}{2\pi} \int_0^{2\pi}\!\!\int_0^L \rho \exp(-\rho^2/2)\,d\rho\,d\theta$$

$$= 1 - \frac{1}{2\pi}\int_0^{2\pi} \exp(-L^2/2)\,d\theta.$$

Making the further transformation $\tan\theta = \frac{B}{A}\tan\varphi$, we have

$$L^2 = A^2 \cos^2\varphi + B^2 \sin^2\varphi,$$
$$d\theta = AB\,d\varphi/L^2,$$

and thus

$$(3.7) \qquad q(A,B) = 1 - \frac{1}{\pi} \int_0^\pi \frac{\exp(-L^2/2)AB\,d\varphi}{L^2} \quad .$$

Now

$$L^2 = A^2 \cos^2\varphi + B^2 \sin^2\varphi = \frac{(A^2 + B^2)}{2} + \frac{(A^2 - B^2)}{2} \cos 2\varphi,$$

so (3.7) becomes

$$(3.8) \qquad q(A,B) = 1 - \exp\left[-\left(\frac{A^2 + B^2}{4}\right)\right]$$

$$\cdot \left\{\frac{1}{\pi} \int_0^\pi \exp\left[\left(-\frac{A^2 - B^2}{4}\right)\cos 2\varphi\right] \frac{AB\,d\varphi}{L^2}\right\} \quad .$$

Further, letting $\mu = (A - B)/(A + B)$, we have

$$\frac{AB}{L^2} = \frac{2AB}{A^2 + B^2 + (A^2 - B^2)\cos 2\varphi} = \frac{1 - \mu^2}{1 + \mu^2 + 2\mu \cos 2\varphi} \quad .$$

Utilizing Lemma 1, we can write

$$(3.9) \qquad \frac{AB}{L^2} = 1 + 2 \sum_{n=1}^\infty (-1)^n \mu^n \cos 2n\varphi$$

$$= 1 + 2 \sum_{n=1}^\infty (-1)^n \left(\frac{A - B}{A + B}\right)^2 \cos 2n\varphi \quad .$$

From Lemma 2 we obtain the expansion

$$(3.10) \qquad \exp\left[-\left(\frac{A^2 - B^2}{4}\right)\cos 2\varphi\right] = I_0\left(\frac{A^2 - B^2}{4}\right)$$

$$+ 2 \sum_{n=1}^\infty (-1)^n I_n\left(\frac{A^2 - B^2}{4}\right)\cos 2n\varphi.$$

Substituting (3.9) and (3.10) in (3.8), and using the fact that

$$\int_0^\pi \cos 2m\varphi \cos 2n\varphi \, d\varphi = \frac{\pi}{2}$$

for $m = n$ and zero otherwise, we obtain

$$(3.11) \qquad q(A,B) = 1 - \exp\left(-\frac{A^2 + B^2}{4}\right)\left[I_0\left(\frac{A^2 - B^2}{4}\right)\right.$$

$$\left. + 2\sum_{n=1}^{\infty} \left(\frac{A-B}{A+B}\right)^n I_n\left(\frac{A^2 - B^2}{4}\right)\right].$$

If we let

$$\frac{A+B}{2} = R, \quad \frac{A-B}{2} = r,$$

it follows that

$$\frac{A^2 + B^2}{4} = \frac{R^2 + r^2}{2} \quad \text{and} \quad \frac{A^2 - B^2}{4} = Rr,$$

so that (3.11) becomes

$$(3.12) \qquad q(A,B) = 1 - \exp\left(-\frac{R^2 + r^2}{2}\right)\left[I_0(Rr) + 2\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n I_n(rR)\right].$$

By Lemma 3 the right-hand side of (3.12) is $p(R,r) - p(r,R)$, so

$$q(A,B) = p(R,r) - p(r,R) = p\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$$

$$- p\left(\frac{A-B}{2}, \frac{A+B}{2}\right).$$

Theorem 2. The integral of an elliptical Gaussian distribution with components of standard deviation $\sigma_x \geq \sigma_y$ and origin zero over a circle of radius R is given by the elliptical coverage function $q\left(\frac{R}{\sigma_y}, \frac{R}{\sigma_x}\right)$.

Proof. The integral J of the elliptical Gaussian distribution over a circle of radius R is given by

$$J = \frac{1}{2\pi} \sigma_x \sigma_y \int\int_C \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma x^2} + \frac{y^2}{\sigma y^2}\right)\right] dx \, dy,$$

where $C: x^2 + y^2 = R^2$. Under the transformation $x/\sigma x = \xi$, $y/\sigma y = \eta$, we obtain

$$J = \frac{1}{2\pi} \int\int_{C'} \exp\left[-(\xi^2 + \eta^2)/2\right] d\xi \, d\eta$$

where $C': \sigma x^2 \xi^2/R^2 + \sigma y^2 \eta^2/R^2 = 1$. Thus, by definition of the elliptic coverage function,

$$J = q\left(\frac{R}{\sigma_y}, \frac{R}{\sigma_x}\right).$$

Using Theorem 1 and the tabulated values of the circular coverage function of Ref. 1, we present in Table 1 a short table of values of the elliptic coverage function $q\left(\frac{R}{\sigma_y}, \frac{R}{\sigma_x}\right)$. For convenience we have used as entries the ratios $\rho = \sigma_y/\sigma_x$ and $R/\sigma_M = R/\sigma_x$, where σ_M is the maximum standard deviation. Thus

$$q\left(\frac{R}{\sigma_y}, \frac{R}{\sigma_x}\right) = q\left(\frac{R}{\rho \sigma_M}, \frac{R}{\sigma_M}\right) = \rho\left[\frac{R}{\sigma_M}\left(\frac{1+\rho}{2\rho}\right), \frac{R}{\sigma_M}\left(\frac{1-\rho}{2\rho}\right)\right]$$

$$- \rho\left[\frac{R}{\sigma_M}\left(\frac{1-\rho}{2\rho}\right), \frac{R}{\sigma_M}\left(\frac{1+\rho}{2\rho}\right)\right].$$

For $\rho = 0$, i.e., $\sigma_y = 0$, the distribution q becomes the linear Gaussian, i.e.,

$$\lim_{\sigma_y \to 0} q\left(\frac{R}{\sigma_y}, \frac{R}{\sigma_x}\right) = \sqrt{\frac{2}{\pi}} \int^{R/\sigma_x} \exp(-x^2/2) \, dx.$$

The data of Table 1 are presented in a very convenient graphical form in Fig. 1.

Table 1

q AS A FUNCTION OF R/σ_M AND ρ

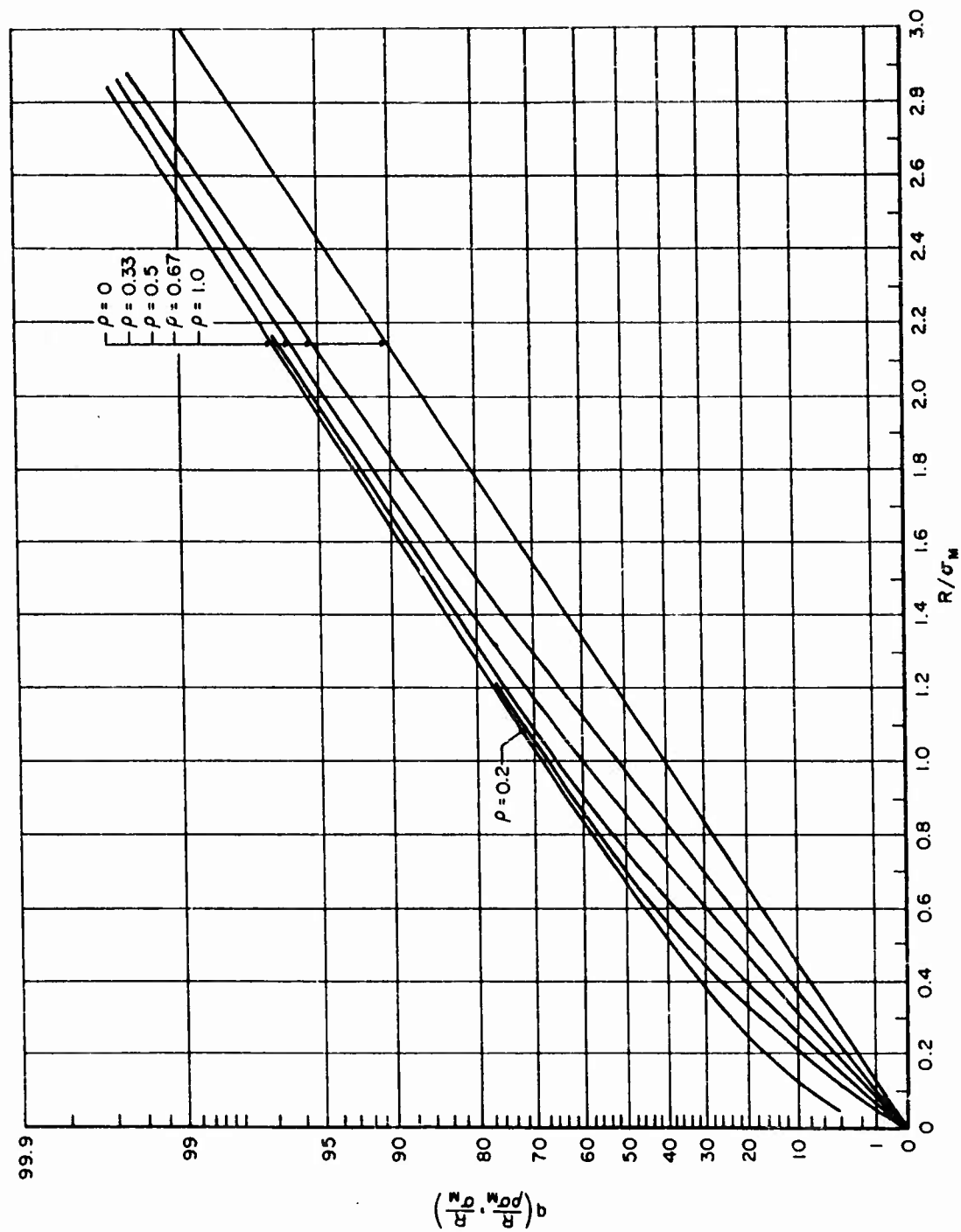| R/σ_M	ρ						
	0	0.2	0.25	0.33	0.5	0.67	1
0	0	0	0	0	0	0	0
0.2	159	088	074	057	039	030	020
0.4	311	264	238	199	145	113	077
0.6	451	435	407	369	292	234	165
0.8	576	560	549	523	448	375	274
1.0	683	673	666	649	590	505	393
1.2	770	763	759	748	707	637	513
1.4	838	834	832	825	799	748	625
1.6	890	888	886	882	866	832	722
1.8	928	926	925	923	913	891	802
2.0	955	953	952	951	945	932	865
2.2	972	971	971	—	966	959	911
2.4	983	983	983	—	981	976	944
2.6	991	—	991	—	989	986	966
2.8	995	—	995	—	994	993	980

Fig. 1—Elliptic coverage function

4. PROBABILITY CIRCLES—CEP AND CIRCULAR QUANTITIES

In Sec. 3 we have determined the probability q in terms of the circle radius R and the standard deviations σ_x and σ_y It is often desirable to determine, for a fixed value of q, the radius R_q of that circle for which the elliptic coverage function (probability of landing within R_q) is q. The more commonly used of these are the 0.5 circle, whose radius $R_{0.5}$ is called the CEP, and the 0.25 and 0.75 circles whose radii we will call the lower and upper circular quartiles, respectively.

Figure 1, which shows q as a function of ρ and R/σ_M, can also be considered to show R/σ_M as a function of q and ρ. Let $h(q,\rho) = R/\sigma_M$. Then

$$R_q = \sigma_M h(q,\rho) ,$$

where h is graphed in Fig. 1. It is often desirable to relate R_q to the arithmetic or quadratic means of the σ's, i.e.,

$$\bar{\sigma} = \frac{\sigma_x + \sigma_y}{2} , \quad Q^2 = \frac{\sigma_x^2 + \sigma_y^2}{2} .$$

In Fig. 2 the ratios of $R_q/\bar{\sigma}$ for q = 0.25, 0.50, 0.75 are plotted against the ratio ρ, while in Fig. 3 the same is done for the ratio R_q/Q.

A particular advantage of the ratio R_q/Q is that the quadratic mean Q (see Appendix) is independent of correlation. Of particular interest in Figs. 2 and 3 are the flatness of the curves $R_{0.5}/\bar{\sigma}$ and $R_{0.75}/Q$, i.e., for q of the order of 0.5,

of the order of 0.75, R/Q is independent of ρ for all ρ. Thus
we have as approximations

$$R_{0.5} = CEP = 1.17 \; \bar{\sigma}, \quad \rho \geq 0.2,$$
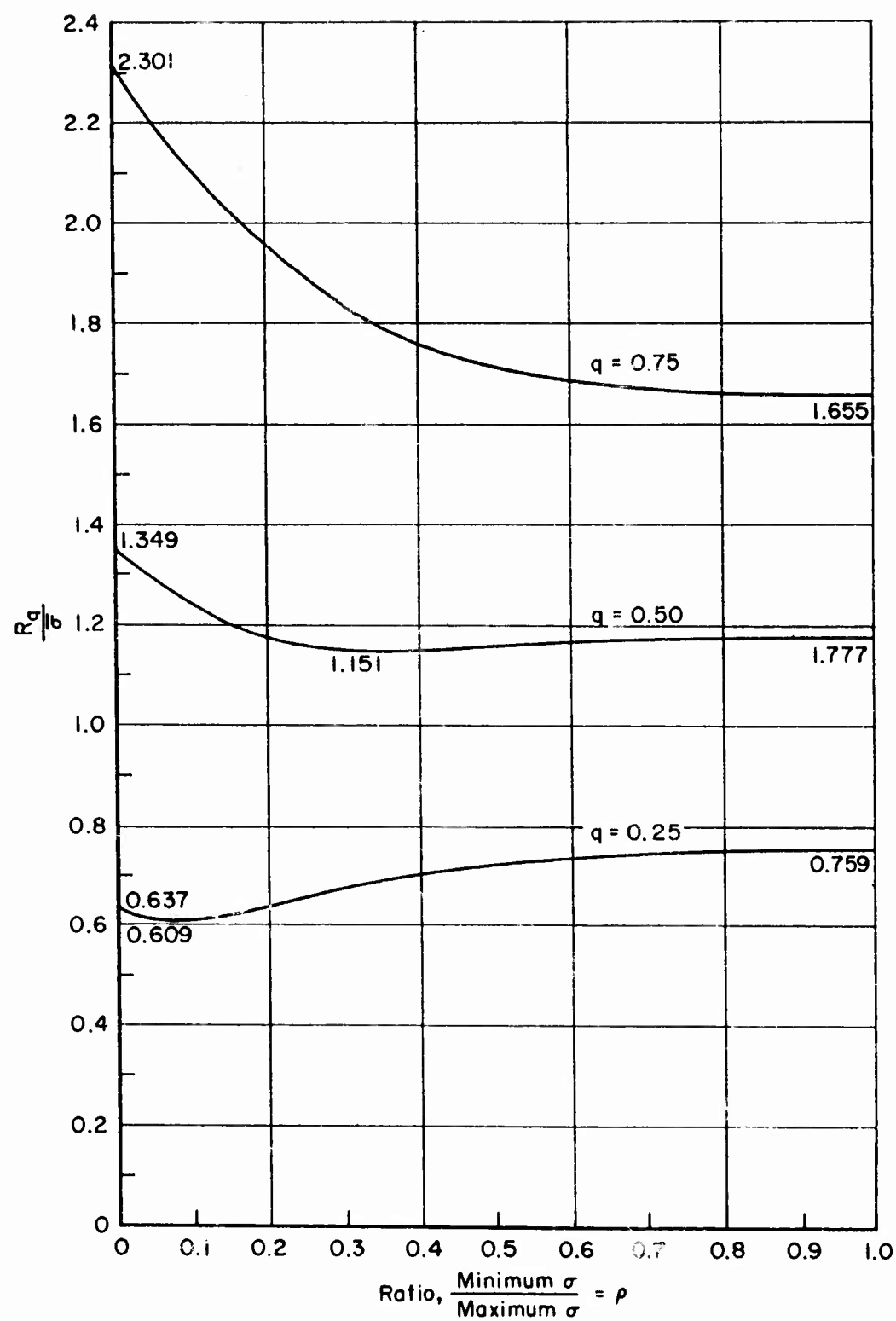
$$R_{0.75} = \text{upper circular quartile} = 1.65 \; Q.$$

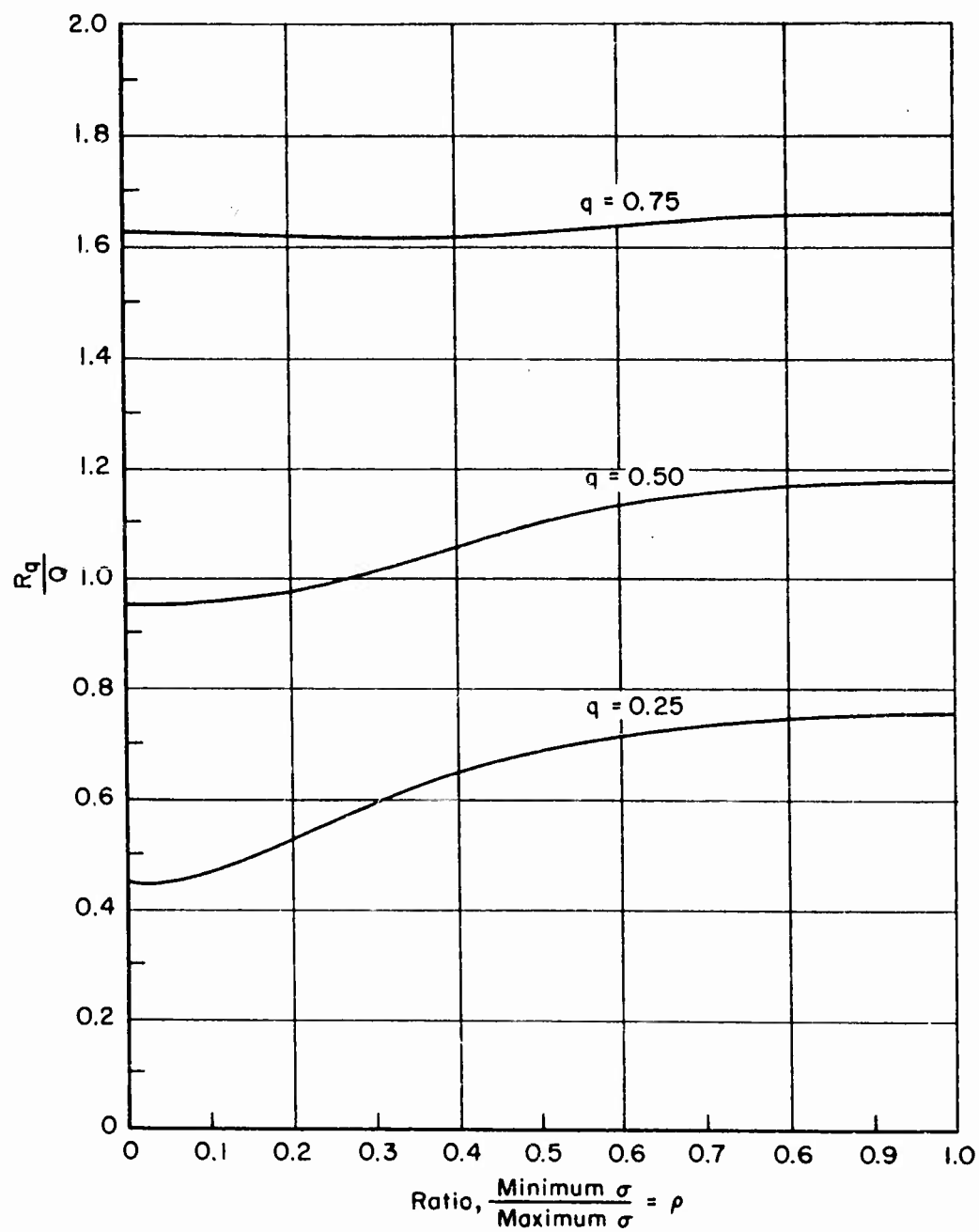Fig. 2 — Ratio $\dfrac{R_q}{\sigma}$ as a function of ρ

Fig. 3 — Ratio $\dfrac{R_q}{Q}$ as a function of ρ

Appendix

THE CORRELATED BIVARIATE DISTRIBUTION

Consider a bivariate Gaussian distribution in the xy—plane, with center at the origin, with components of standard deviation equal to σ_x and σ_y, and with a coefficient of correlation equal to r. The equation of such a distribution can be written

$$dp = \frac{d_x d_y}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left[-\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)/2\,(1-r^2)\right].$$
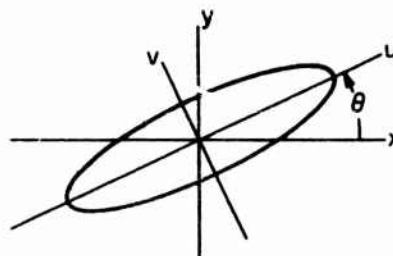

Fig. 4

The substitution

$$\begin{cases} x = u\cos\theta - v\sin\theta \\ y = u\sin\theta + v\cos\theta, \end{cases}$$

where

$$\tan 2\theta = \frac{2r\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2} \ ,$$

rotates the axes and converts this into an uncorrelated, bivariate Gaussian distribution:

$$dp = \frac{d_u d_v}{2\pi\sigma_u\sigma_v} \ \exp\left[-\ (\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2})/2\right].$$

The components, σ_u and σ_v, of standard deviation along the new axes are given by

$$\sigma_u^2 = \tfrac{1}{2}(\sigma_x^2 + \sigma_y^2 + c^2)$$

and

$$\sigma_v^2 = \tfrac{1}{2}(\sigma_x^2 + \sigma_y^2 - c^2),$$

where

$$c^2 = \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + (2r\sigma_x\sigma_y)^2} \ .$$

It should be observed in passing that

$$\sigma_u^2 + \sigma_v^2 = \sigma_x^2 + \sigma_y^2 \ ,$$

regardless of the value of θ (and hence regardless of the value of r).

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